

A group-invariant version of Lehmer's conjecture on heights

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Abstract

We state and prove a group-invariant version of Lehmer's conjecture on heights, generalizing papers by Zagier (1993) [5] and Dresden (1998) [1] which are special cases of this theorem. We also extend their three cases to a full classification of all finite cyclic groups satisfying the condition that the set of all orbits for which every non-zero element lies on the unit circle is finite and non-empty.

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1. A Lehmer-type problem for the Weil height

The *Mahler measure* of a non-zero polynomial $f \in \mathbb{Z}[x]$ given by

$$f(x) = a_n \prod_{i=1}^n (x - \alpha_i) \quad (1)$$

is defined as

$$M(f) = |a_n| \prod_{i=1}^n \max(|\alpha_i|, 1).$$

In 1933, Lehmer asked whether there exists a lower bound $D > 1$ such that for all $f \in \mathbb{Z}[x]$ it holds that

$$M(f) = 1 \quad \text{or} \quad M(f) \geq D.$$

He showed that if such a D exists, then $D \leq 1.1762808\dots$, the largest real root of the polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ [4]. Nowadays, this is still the smallest known value of $M(f) > 1$ for $f \in \mathbb{Z}[x]$.

Mahler's measure is related to the Weil height of an algebraic number. Let K be an algebraic number field and v a place of K . We assume that this v -adic

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valuation is normalized in such a way that for all non-zero $\alpha \in K$ the product of $|\alpha|_v$ over all places v is equal to 1 and the product of $|\alpha|_v$ over all Archimedean v is equal to the absolute value of $N_{K/\mathbb{Q}}(\alpha)$. Then, for $\alpha \in K^*$ the (logarithmic) Weil height h is defined by

$$h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_v \log^+ |\alpha|_v,$$

where the sum is over all places v of K . We used the notation $\log^+(z)$ to refer to $\log \max(z, 1)$ for $z \in \mathbb{R}$. The Weil height is independent of K and if the polynomial (1) is the minimal polynomial of α over \mathbb{Q} , then $h(\alpha) = \frac{1}{n} \log M(f)$. This Weil height can be extended to $\mathbb{P}^1(\overline{\mathbb{Q}})$. Namely, for $x = [x_1 : x_2] \in \mathbb{P}^1(\overline{\mathbb{Q}})$, we define

$$h(x) = \frac{1}{[K : \mathbb{Q}]} \sum_v \log \max(|x_1|_v, |x_2|_v),$$

where K is chosen such that $x_1, x_2 \in K$. Note that $h(x) \geq 0$.

Definition 1. Let G be a finite subgroup of $\mathrm{PGL}_2(\mathbb{Q})$. The G -orbit height of $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$ is defined as

$$h_G(x) = \sum_{\sigma \in G} h(\sigma x).$$

Note that $h_G(x) \geq 0$ and $h_G(\sigma\alpha) = h_G(\alpha)$ for all $\sigma \in G$. We can now state the G -invariant Lehmer problem, namely: given a finite group G does there exist a positive lower bound D such that

$$h_G(x) = 0 \quad \text{or} \quad h_G(x) \geq D \tag{2}$$

for all $x \in \mathbb{P}^1(\overline{\mathbb{Q}})$? As Zagier pointed out [5], if G is trivial such a constant does not exist (e.g., $h_{\{e\}}(\sqrt[n]{2}) = n^{-1} \log 2 \rightarrow 0$). Assuming a mild restriction on G , which we will state next, we will prove that this lower bound D exists for h_G .

Recall that as a consequence of Kronecker's lemma [3], for $\alpha \in K$ we have that $h(\alpha) = 0$ if and only if $\alpha = 0$ or α is a root of unity. We will now define a set \mathcal{O} of orbits such that elements of these orbits are precisely the zeros of h_G over K .

Definition 2. Let \mathcal{Q} be the set of all orbits of the action of G on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\mathcal{O} \subset \mathcal{Q}$ be the set of all orbits for which every non-zero element lies on the unit circle, i.e.

$$\mathcal{O} = \{O \in \mathcal{Q} \mid \forall z \in O : z = 0 \text{ or } |z| = 1\}.$$

The main purpose of this note is to solve the G -invariant Lehmer problem in the case that h_G has finitely many zeros:

Theorem 1. *If \mathcal{O} is finite, then there exists a positive D such that*

$$h_G(\alpha) = 0 \quad \text{or} \quad h_G(\alpha) \geq D$$

for all $\alpha \in K$.

Remark 1. For a finite $G \leq \mathrm{PGL}_2(\mathbb{Q})$ this theorem can also be stated in terms of Mahler measures instead of heights. Let α be a given algebraic integer with minimal polynomial $f \in \mathbb{Z}[x]$ of degree n . Assume $\sigma \in G$ and write $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let

$$f_\sigma(z) = C(cz + d)^n f(\sigma(z)),$$

where $C \in \mathbb{Q}$ is chosen in such a way that f_σ is primitive, so that f_σ is the minimal polynomial of $\sigma^{-1}\alpha$. Then, this theorem implies that there exists a constant $E > 1$ such that

$$\prod_{\sigma \in G} M(f_\sigma) = 1 \quad \text{or} \quad \prod_{\sigma \in G} M(f_\sigma) \geq E^n$$

for all primitive irreducible polynomials $f \in \mathbb{Z}[x]$.

Later we will see that if \mathcal{O} is infinite, its subsets contain all roots of unity. Moreover, if \mathcal{O} is finite, its subsets can only contain 0 and the roots of cyclotomic polynomials of degree at most 2. For the cyclic case $G = \langle \sigma \rangle$, we will use this to classify all $\sigma \in \mathrm{PGL}_2(\mathbb{Q})$ for which \mathcal{O} is finite and non-empty. In nearly all cases, it is also possible to calculate the maximal value of D for which Theorem 1 holds. For three of these cases, these values are already known. By a theorem of Zhang, for which Zagier gave an elementary proof, we have that $D = \frac{1}{2} \log \frac{1+\sqrt{5}}{2} = 0.2406059\dots$ for $G = \{z, 1-z\}$ [6, 5]. Here we identified $\mathrm{PGL}_2(\mathbb{Q})$ with the Möbius transformations, maps $\sigma : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of the form $\sigma(z) = \frac{az+b}{cz+d}$ for $a, b, c, d \in \mathbb{Q}$. Dresden proved that $D = \log |\beta| = 0.4217993\dots$ for $G = \{z, \frac{1}{1-z}, 1 - \frac{1}{z}\}$, where β is a maximal root in absolute value of $(z^2 - z + 1)^3 - (z^2 - z)^2$ and mentioned that for $G = \{z, \frac{z+1}{-z+1}, -\frac{1}{z}, \frac{z-1}{z+1}\}$ one has $D = \log |\gamma| = 0.7328576\dots$ for γ a maximal root in absolute value of $(z^2 + 1)^4 + z^2(z^2 - 1)^2$ [1].

2. Proof of Theorem 1

Set $\mathcal{O} = \{O_i \mid i \in \{1, 2, \dots, k\}\}$. For each orbit $O_i \in \mathcal{O}$ we choose $\alpha_i \in O_i$ and define $p_i \in \mathbb{Z}[x]$ as the minimal polynomial of α_i . Let N be the maximum of the degrees of all the p_i and let

$$n_v = \begin{cases} 0 & \text{if } v \text{ is non-Archimedean,} \\ 1 & \text{if } v \text{ is real,} \\ 2 & \text{if } v \text{ is complex.} \end{cases}$$

The proof of Theorem 1 will follow directly from the following lemma:

Lemma 2. *Let v be a place of K and $\alpha \in K$. There exists $B_{\max} > 0$ such that for all B with $0 < B < B_{\max}$ there exists a positive D such that*

$$\sum_{\sigma \in G} \left(\log^+ |\sigma(\alpha)|_v - \frac{1}{2} \log |\sigma(\alpha)|_v - B \sum_{i=1}^k \log |p_i(\sigma(\alpha))|_v \right) \geq n_v D. \quad (3)$$

Proof. Firstly, assume v is finite. We will show that the summand

$$\log^+ |\sigma(\alpha)|_v - \frac{1}{2} \log |\sigma(\alpha)|_v - B \sum_{i=1}^k \log |p_i(\sigma(\alpha))|_v \quad (4)$$

of (3) is nonnegative for all $\sigma \in G$. If $\sigma(\alpha)$ is integral at v , it follows that $\log^+ |\sigma(\alpha)|_v = 0$ and $\frac{1}{2} \log |\sigma(\alpha)|_v \leq 0$. By writing $p_j(\sigma(\alpha)) = b_{jn}\sigma(\alpha)^n + \dots + b_{j0}$ for $j \in \{1, 2, \dots, k\}$ and $b_{ji} \in \mathbb{Z}$ we obtain

$$|p_j(\sigma(\alpha))|_v \leq \max(|b_{jn}|_v \cdot |\sigma(\alpha)|_v^n, \dots, |b_{j0}|_v) \leq 1.$$

Therefore, $\sum_{i=1}^k \log |p_i(\sigma(\alpha))|_v \leq 0$, which implies that the summand (4) is nonnegative for all $B \in \mathbb{R}^+$.

If $|\sigma(\alpha)|_v > 1$, we find that $\log^+ |\sigma(\alpha)|_v - \frac{1}{2} \log |\sigma(\alpha)|_v = \frac{1}{2} \log |\sigma(\alpha)|_v > 0$. Using the same notation as above,

$$\log |p_j(\sigma(\alpha))|_v \leq \log \max(|b_{jn}|_v \cdot |\sigma(\alpha)|_v^n, \dots, |b_{j0}|_v) \leq N \log |\sigma(\alpha)|_v.$$

Therefore, for

$$B_{\max} \leq \frac{1}{2kN} \leq \frac{\log |\sigma(\alpha)|_v}{2 \sum_{i=1}^k \log |p_i(\sigma(\alpha))|_v}$$

the summand (4) is positive for all B with $0 < B < B_{\max}$ and all $\sigma \in G$.

Secondly, if v is Archimedean, then $|\alpha|_v = |\iota(\alpha)|^{n_v}$ for some embedding ι of K into \mathbb{C} . Let

$$g_1(z) = \sum_{\sigma \in G} (\log^+ |\sigma(z)| - \frac{1}{2} \log |\sigma(z)|) \quad \text{and} \quad g_2(z) = - \sum_{\sigma \in G} \sum_{i=1}^k \log |p_i(\sigma(z))|.$$

The claim is that for $z \in \mathbb{C}$ the function

$$f(z) = g_1(z) + B \cdot g_2(z) = \sum_{\sigma \in G} \left(\log^+ |\sigma(z)| - \frac{1}{2} \log |\sigma(z)| - B \sum_{i=1}^k \log |p_i(\sigma(z))| \right)$$

is bounded below by some constant $D > 0$. Clearly, f tends to infinity as $\sigma(z)$ tends to zero or to one of the roots of the p_i . As $\sum_{i=1}^k \log |p_i(\sigma(z))| \leq k \log |C\sigma(z)^N|$ for some $C \in \mathbb{R}^+$ and $\sigma(z)$ sufficiently large, assuming $B_{\max} < \frac{1}{2kN}$ we find that if $\sigma(z)$ tends to infinity then f tends to infinity. As f is continuous elsewhere and harmonic if $|\sigma(z)| \neq 1$ for all $\sigma \in G$, it attains a minimum on a circle $|\sigma(z)| = 1$ for some $\sigma \in G$. As $f(z) = f(\sigma(z))$ for all $\sigma \in G$, we can assume that this minimum is attained on the unit circle. If this minimum is strictly positive, we are done. Otherwise, $g_2(z) \leq 0$ and this minimum is attained in the set $S = \{z \in \mathbb{C} \mid |z| = 1 \text{ and } g_2(z) \leq 0\}$. Let $q \in S$ and let Q be the orbit of q . If $Q \in \mathcal{O}$, write $Q = \mathcal{O}_i$. Then, there is a $\tau \in G$ such that $\tau(q)$ is a root of p_i . It follows that g_2 tends to infinity as z tends to q , contradicting $q \in S$. Therefore, $Q \notin \mathcal{O}$. Hence, there exists a $\tau \in G$ such that $\tau(q) \neq 0$ and $|\tau(q)| \neq 1$. This implies that $\log^+ |\tau(q)| - \frac{1}{2} \log |\tau(q)| > 0$. As for

all $z \in \mathbb{C}$ we have that $\log^+ |z| - \frac{1}{2} \log |z| \geq 0$, it follows that $g_1(q) > 0$ for all $q \in S$. As S is compact, g_1 attains a minimum $m > 0$ in S . Also, g_2 attains a minimum n in S . Letting $B_{\max} < -m/n$, it follows that f attains a positive minimum D in S . \square

Proof of Theorem 1. Observe that for $\beta \in K^*$ we have

$$\sum_v n_v = [K : \mathbb{Q}] \quad \text{and} \quad \sum_v \log |\beta|_v = 0. \quad (5)$$

Then, for α for which there is no $\sigma \in G$ such that $\sigma(\alpha)$ is zero, infinite or a root of some p_i , we can sum the inequality (3) in Lemma 2 over all places v of K and apply (5). After dividing by $[K : \mathbb{Q}]$ we find that $h_G(\alpha) \geq D$ for all but finitely many $\alpha \in K$. Hence, it follows that for some possibly smaller value of D and all $\alpha \in K$ we have $h_G(\alpha) = 0$ or $h_G(\alpha) \geq D$. \square

3. When is \mathcal{O} finite?

We will investigate how strong the condition is that \mathcal{O} is finite.

Proposition 3. *The set \mathcal{O} is finite if and only if there exists a root of unity ζ such that $h_G(\zeta) > 0$.*

Proof. If \mathcal{O} is finite, we can choose a root of unity ζ and $\tau \in G$ with $|\tau\zeta| \neq 0, 1$. As $h(\sigma\zeta) \geq 0$ for all $\sigma \in G$ and $h(\tau\zeta) > 0$, we find $h_G(\zeta) > 0$.

Conversely, if there exists a root of unity ζ with $h_G(\zeta) > 0$, then there exists a $\tau \in G$ such that $\tau\zeta$ is not a root of unity. It is known that Möbius transformations map real circles on $\hat{\mathbb{C}}$ to real circles on $\hat{\mathbb{C}}$ provided that we regard a line through ∞ as a circle. Hence, τ maps the unit circle to another circle. As two circles intersect in at most two different points, there are at most two roots of unity η such that $\tau\eta$ is also a root of unity. Hence, \mathcal{O} is finite. \square

Corollary 4. *Let $G \leq \text{PGL}_2(\mathbb{Q})$ be finite. Then, \mathcal{O} is infinite if and only if G is a subgroup of*

$$\left\{ I, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}, \begin{pmatrix} b & a \\ -a & -b \end{pmatrix} \right\} \quad (6)$$

for some $a, b \in \mathbb{Q}$ with $a^2 \neq b^2$.

Proof. If \mathcal{O} is infinite then $\sigma(1) = \pm 1$ and $\sigma(-1) = \mp 1$. Hence, an element $\sigma \in G$ is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ -b & -a \end{pmatrix}$$

for some $a, b \in \mathbb{Q}$ with $a^2 \neq b^2$. The first is of infinite order unless $a = 0$ or $b = 0$. The product of two elements

$$\begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a' & b' \\ -b' & -a' \end{pmatrix}$$

of $\mathrm{PGL}_2(\mathbb{Q})$ is of finite order if and only if $ab' = ba'$ or $aa' = bb'$. Hence, G must be a subgroup of (6). It can easily be checked that \mathcal{O} is infinite for subgroups of (6). \square

4. When is the G -orbit height zero?

Denote with $\pm\frac{1}{2} \pm' \frac{1}{2}i\sqrt{3}$ the four primitive third and sixth roots of unity, where the sign \pm' can be chosen independently from the sign \pm .

Lemma 5. *Let \mathcal{O} be finite and $G \leq \mathrm{PGL}_2(\mathbb{Q})$. Then $h_G(\alpha) = 0$ if and only if α equals $0, \pm 1, \pm i$ or $\pm\frac{1}{2} \pm' \frac{1}{2}i\sqrt{3}$.*

Proof. If $h_G(\alpha) = 0$, we have for all $\sigma \in G$ that $\sigma\alpha$ equals 0 or is a root of unity. Assuming $\alpha \in \mathbb{Q}$, we find α equals 0 or ± 1 . If $\alpha \notin \mathbb{Q}$, then for all $\sigma \in G$ we also have $\sigma\alpha \notin \mathbb{Q}$, so $\sigma\alpha$ is a root of unity. By the proof of the previous proposition there is a $\tau \in G$ which maps at most two roots of unity to other roots of unity. As $h_G(\alpha) = h_G(\alpha')$ for all algebraic conjugates α' of α , it follows that the minimal polynomial of α has degree at most two. Hence, α equals $\pm i$ or $\pm\frac{1}{2} \pm' \frac{1}{2}i\sqrt{3}$. \square

5. Explicit constants

Dresden proved that all finite subgroups of $\mathrm{PGL}_2(\mathbb{Q})$ are isomorphic to the cyclic group C_n or the dihedral group D_n (where the latter is of order $2n$) for $n = 1, 2, 3, 4$ or 6 [2]. By the previous lemma, there are only 9 possible elements in orbits in \mathcal{O} . Hereby, it is possible to determine all finite cyclic groups $G \leq \mathrm{PGL}_2(\mathbb{Q})$ for which \mathcal{O} is finite and non-empty. Moreover, by generalizing Zagier's and Dresden's proofs [5, 1], it is possible to explicitly find the best value of D in Theorem 1. We have collected these data in Table 1, meaning the following: the first column of this table gives a list of one generator $\sigma \in \mathrm{PGL}_2(\mathbb{Q})$ for every cyclic group $G = \langle \sigma \rangle$ satisfying the condition that \mathcal{O} is finite and non-empty. It is assumed that $p, q \in \mathbb{Z}$ are relatively prime such that $q > 0$, $\det(\sigma) \neq 0$, $p/q \neq 0$ in the first and fifth row and $p/q \neq \mp 1/2$ in the seventh row. The second column shows the elements of \mathcal{O} , where we use the shorthand notation $\omega_{\pm\pm'} = \pm\frac{1}{2} \pm' \frac{1}{2}i\sqrt{3}$. The third column shows the order of σ . The remaining columns give information needed to determine the optimal value of D . We let $\phi(z) = \frac{1}{E} \prod_{\sigma \in G} p_1(\sigma(z))$ where p_1 corresponds to an orbit in \mathcal{O} as in the proof of Theorem 1 and E is such that the numerator and denominator of

$\phi(z)$ are relatively prime. The inequality $\sum_{i=0}^{\mathrm{ord} \sigma - 1} \log^+ |\sigma^i(z)| - B \log |\phi(z)| \geq D$

holds where the values of B and $\exp(D)$ can be found in the corresponding row of the columns ' B ' and ' $\exp(D)$ '. From this, in a similar fashion as the proof of Theorem 1, one deduces $h_G(\alpha) = 0$ or $h_G(\alpha) \geq D$ for all $\alpha \in \overline{\mathbb{Q}}$. The element α in the last column is such that equality holds in $h_G(\alpha) \geq D$. However, it is not unique. Here, a maximal root of a polynomial is defined as a root which

Table 1: Classification of cyclic groups with \mathcal{O} non-empty and finite, together with data to find the optimal constant D of Theorem 1. In Section 5 the meaning of these data is explained.

σ	Elts. of \mathcal{O}	ord σ	B	$\exp(D)$	$D \approx$ (for $p/q=5$)	Equality
$\begin{pmatrix} 1 & 0 \\ p/q & -1 \end{pmatrix}$	$\{0\}$	2	1	$\max(p - q, q)$	1.38629	$\begin{cases} 1 & \text{if } p/q > 0 \\ -1 & \text{if } p/q < 0 \end{cases}$
$\begin{pmatrix} 1 & \pm 1 \\ p/q & -1 \end{pmatrix}$	$\{0, \mp 1\}$	2	1	$\max(p \mp q /2, q)$	0.69315	$\pm 1 \quad \text{if } 2 \mid p \mp q$
$\begin{pmatrix} 1 & p/q \\ p/q \pm 2 & -1 \end{pmatrix}$	$\{\pm 1\}$	2	1	$\max(p \pm 3q /4, p \mp q /4)$	0.69315	$\mp 1 \quad \text{if } 4 \mid p \mp q$
$\begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}$	$\{0, -1, 1\}$	3	1	5	1.60944	i
$\begin{pmatrix} 1 & p/q \\ p/q & -1 \end{pmatrix}$	$\{i, -i\}$	2	$\frac{1}{2}$	$ p + q/2$	1.09861	$1, -1 \quad \text{if } 2 \mid p + q$
[1] $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$	$\{i\}, \{-i\}$	4	0.19408...	$ \alpha $	0.73286	α , maximal root of $(z^2 + 1)^4 + z^2(z^2 - 1)^2$
$\begin{pmatrix} 1 & p/q \\ p/q \pm 1 & -1 \end{pmatrix}$	$\{\omega_{\pm+}, \omega_{\pm-}\}$	2	$\frac{1}{2}$	$\max(p \pm 2q /3, p \mp q /3)$	0.84730	$\mp 1 \quad \text{if } 3 \mid p \mp q$
[1] $\begin{pmatrix} 0 & 1 \\ -1 & \pm 1 \end{pmatrix}$	$\{\omega_{\pm+}\}, \{\omega_{\pm-}\}$	3	0.11724...	$ \alpha $	0.42180	α , maximal root of $(z^2 \mp z + 1)^3 - (z^2 \mp z)^2$
$\begin{pmatrix} 2 & \mp 1 \\ \pm 1 & 1 \end{pmatrix}$	$\{\omega_{\pm+}\}, \{\omega_{\pm-}\}$	6	0.30503...	$\left \frac{\alpha(2\alpha-1)}{\alpha+1} \right $	1.75737	α , maximal root of $(z^2 \mp z + 1)^6 + z^2(2z^2 \mp 5z + 2)^2(z^2 - 1)^2$

Table 2: Cyclic groups $\langle \sigma \rangle$ with \mathcal{O} non-empty and finite which appear twice in Table 1, together with data to find the optimal constant D of Theorem 1. In Section 5 the meaning of these data is explained.

	σ	Elt. of \mathcal{O}	B_1	B_2	$\exp(D)$	$D \approx$	Equality
	$\begin{pmatrix} 1 & 0 \\ \pm 1 & -1 \end{pmatrix}$	$\{0\}, \{\omega_{\pm\pm}, \omega_{\pm-}\}$	1/2	1/2	$\sqrt{\frac{1+\sqrt{5}}{2}}$	0.24061	$\mp e^{\frac{2\pi i}{5}}$
	$\begin{pmatrix} 1 & 0 \\ \pm 2 & -1 \end{pmatrix}$	$\{0\}, \{\pm 1\}$	2/3	1/3	$\sqrt{3}$	0.54931	$\omega_{\pm\pm}$
[5, 6]	$\begin{pmatrix} 1 & \mp 1 \\ 0 & -1 \end{pmatrix}$	$\{0, \pm 1\}, \{\omega_{\pm\pm}, \omega_{\pm-}\}$	$\frac{\sqrt{5}-1}{2\sqrt{5}}$	$\frac{1}{4\sqrt{5}}$	$\sqrt{\frac{1+\sqrt{5}}{2}}$	0.24061	$\pm e^{\frac{\pi i}{5}}$
	$\begin{pmatrix} 1 & \mp 1 \\ \mp 1 & -1 \end{pmatrix}$	$\{0, \pm 1\}, \{i, -i\}$					
	$\begin{pmatrix} 1 & \mp 1 \\ \mp 2 & -1 \end{pmatrix}$	$\{0, \pm 1\}, \{\omega_{\mp+}, \omega_{\mp-}\}$					
	$\begin{pmatrix} 1 & \mp 1 \\ \mp 3 & -1 \end{pmatrix}$	$\{0, \pm 1\}, \{\mp 1\}$					

is maximal in absolute value and which imaginary part is nonnegative. For the polynomials we apply this definition to, this uniquely determines the root.

Example 1. Consider the second row of Table 1 and choose \pm to be $+$, that is $\sigma(z) = \frac{z+1}{p/q \cdot z - 1}$ and $p/q \neq -1$. Then, $\mathcal{O} = \{\{0, -1\}\}$, $\phi(z) = \frac{z(z+1)}{pz-q}$ and $D = \log(\max(|p-q|/2, |q|))$. Then

$$\log^+ |z| + \log^+ |\sigma(z)| - \log |\phi(z)| \geq D,$$

which yields $h_G(0) = 0$, $h_G(-1) = 0$ and $h_G(\alpha) \geq D$ for all $\alpha \in \overline{\mathbb{Q}}$ with $\alpha \neq 0, -1$. We have $h_G(\alpha) = D$ for $\alpha = 1$ if $2 \mid p-q$.

There are $\sigma \in \text{PGL}_2(\mathbb{Q})$ which can be found twice in Table 1. For these σ we have that the corresponding value of D is not positive, so this value of D is not allowed in Theorem 1. All these σ can be found in Table 2. They all have order 2. Using another inequality, it is in some cases still possible to find the optimal (positive) value of D . Namely, it holds that $\log^+ |z| + \log^+ |\sigma(z)| - B_1 \log |\phi_1(z)| - B_2 \log |\phi_2(z)| \geq D$ for corresponding values in the columns ' B_1 ', ' B_2 ' and ' $\exp(D)$ '. Here, ϕ_1 and ϕ_2 are similarly defined, that is, $\phi_i(z) = \frac{1}{E_i} \prod_{\sigma \in G} p_i(\sigma(z))$ where p_i corresponds to the i th orbit in \mathcal{O} as in the proof of Theorem 1 and E_i is such that the numerator and denominator of $\phi_i(z)$ are relatively prime. Again, it follows that $h_G(\alpha) = 0$ or $h_G(\alpha) \geq D$ for all $\alpha \in \overline{\mathbb{Q}}$. In three of the cases, the author was not able to find the optimal value of D with corresponding values of B_1 and B_2 .

6. Generalizations

It is possible to extend Table 1 and Table 2 to non-cyclic subgroups of $\mathrm{PGL}_2(\mathbb{Q})$. For example consider

$$G = \left\langle \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle \simeq D_3,$$

the dihedral group with 6 elements. Then $\mathcal{O} = \{\{-1, 0, 1\}\}$ and one can show that $h_G(\alpha) \geq \log(25)$ for $\alpha \neq -1, 0, 1$ with equality for $\alpha = i$. Similarly, for

$$G = \left\langle \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \right\rangle \simeq D_2,$$

one has $\mathcal{O} = \{\{\omega_{++}, \omega_{+-}\}\}$ and one can show that $h_G(\alpha) \geq \log(2)$ for $\alpha \neq \omega_{++}, \omega_{+-}$ with equality for $\alpha = -1$. It would be interesting to specify for which other subgroups one can find a similar statement.

Another way to extend these tables, is by considering finite $G \leq \mathrm{PGL}_2(\overline{\mathbb{Q}})$. For example, $\sigma(z) = \frac{z-\sqrt{3}}{\sqrt{3}z+1}$ has order 3 and one finds $\mathcal{O} = \{\{i\}, \{-i\}\}$ for $G = \langle \sigma \rangle$. Although the value of D is computed in some cases, this note does not explain how G determines the value of D . It would be interesting to find a universal lower bound on D or to strengthen Theorem 1 by proving that D is greater than some invariant depending on G .

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